Name of college - S.S. College, J-Bad

Deptt - Mathematics

Topic - Continuity and Differentiability
(Real Analysis)

TEACHER'S Name - Dr. Shree Nath Sharma

Time - 9:00 A.M 10:30 A.M

Date - 19-04-2021

B.Sc II (HONS + Sub)

Problem based on Differentiability and Continuity.

<u>Problem</u> :→ A Function $f(x)$ is defined as

$f(x) = 1 + \sin x \qquad 0 \le x < \pi/2$

$f(x) = 2 + (x - \pi/2)^2 \qquad \pi/2 \le x < \infty$

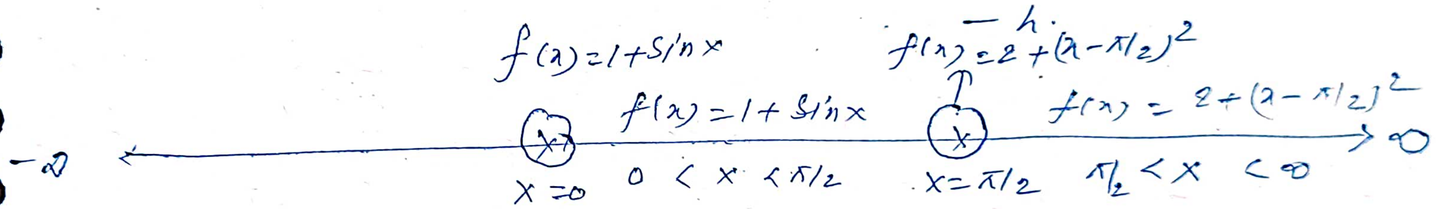
Discuss its differentiability at $x = \pi/2$

<u>Solution</u> :→ $f(x)$ is differentiable at $x = \pi/2$ if $Rf'(\pi/2)$ and $Lf'(\pi/2)$ does Exist and are equal.

<u>Where</u>

$$Rf'(\tfrac{\pi}{2}) = \lim_{h \to 0} \frac{f(\pi/2 + h) - F(\pi/2)}{h}.$$

$$Lf'(\pi/2) = \lim_{h \to 0} \frac{F(\pi/2 - h) - F(\pi/2)}{-h}.$$



for $Rf'(\pi/2)$

$f(\pi/2) = 2 + (\pi/2 - \pi/2)^2 = 2$

$f(\pi/2 + h) = 2 + (\pi/2 + h - \pi/2)^2$

$= 2 + h^2$

$$\therefore Rf'(2) = \lim_{h \to 0} \frac{f(\pi/2 + h) - F(\pi/2)}{h} \quad \text{—— (I)}$$

$$= \lim_{h \to 0} \frac{\cancel{2} + h^2 - \cancel{2}}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0$$

$$Lf'(2) = \lim_{h \to 0} \frac{f(\pi/2 - h) - F(\pi/2)}{-h} \quad \text{—— (II)}$$

$f(\pi/2 - h) = 1 + \sin(\pi/2 - h) = 1 + \cos h$

$f(\pi/2) = 2 + (\pi/2 - \pi/2)^2 = 2$

$$\therefore Lf'(2) = \lim_{h \to 0} \frac{1+\cos h - 2}{-h}$$

$$= \lim_{h \to 0} \frac{-1+\cos h}{-h}$$

$$= \lim_{h \to 0} \frac{-(1-\cos h)}{-h} = \lim_{h \to 0} \frac{1-\cos h}{h}$$

$$= \lim_{h \to 0} \frac{2\sin^2 h/2}{h} = 2 \lim_{h \to 0} \frac{\left(\frac{\sin h/2}{h/2}\right)^2 \times \frac{h^2}{4}}{h}$$

$$= 2 \lim_{h \to 0} \frac{h}{4} = 0$$

Here we see $Rf'(\pi/2)$ and $Lf'(\pi/2)$ both exist and are equal.

Therefore $f(x)$ is differentiable at $x = \pi/2$

2. If $f(x) = \dfrac{x}{1+e^{1/x}}$ $x \neq 0$.

and $f(x) = 0$ when $x = 0$

Show that $f(x)$ is not differentiable ~~and~~ at $x = 0$

Solution :→ By the question

$$f(x) = \frac{x}{1+e^{1/x}} \quad x \neq 0$$

$$f(x) = 0 \quad \text{when } x = 0$$

$f(x)$ is differentiable at $x = 0$

if $Rf'(0)$ and $Lf'(0)$ both exist and are equal

Where $Rf'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

and $Lf'(0) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

Now $f(0+h) = \dfrac{h}{1+e^{1/h}}$ $\quad f(0) = 0$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h - 0}{(1+e^{1/h})\cdot h}$$

$$= \lim_{h\to 0} \frac{h}{(1+e^{1/h})\cdot h} = \lim \frac{1}{1+e^{\infty}} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\dfrac{-h}{1+e^{-1/h}} - 0}{-h}$$

$$= \lim_{h\to 0} \frac{1}{1+e^{-1/h}}$$

$$= \lim_{h\to 0} \frac{1}{1+\frac{1}{e^{h}}}$$

$$= \lim_{h\to 0} \frac{1}{1+\frac{1}{e^{\infty}}}$$

$$= \lim_{h\to 0} \frac{1}{1+\frac{1}{\infty}}$$

$$= \lim_{h\to 0} \frac{1}{1+0} = 1$$

Here we see $Rf'(0)$ and $Lf'(0)$ both exists but are not equal

$\Rightarrow f(x)$ is not differentiable at $x = 0$

Problem 3. If $f(x) = x \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ when $x \neq 0$

$= 0$ when $x = 0$

Show that $f(x)$ is not differentiable at $x = 0$

Solution :→ Here. By the question

$$f(x) = x \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \quad \text{when } x \neq 0$$

$$f(x) = 0 \quad \text{when } x = 0$$

$f(x)$ is differentiable at $x = 0$.

if $Rf'(0)$ and $Lf'(0)$ does Exist and equal

Where $Rf'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$Lf'(0) = \lim\limits_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$$Rf'(0) = \lim_{h \to 0} h \frac{e^{1/h} - e^{-1/h}}{h\,[e^{1/h} + e^{-1/h}]}$$

$$= \lim_{h \to 0} \frac{e^{1/h}(1 - e^{-2/h})}{e^{h}(1 + e^{-2/h})}$$

$$= \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1 - 0}{1 + 0} = 1$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-h\,[e^{-1/h} - e^{1/h}]}{-h\,[e^{-1/h} + e^{1/h}]}$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0 - 1}{0 + 1} = -1$$

Here $Lf'(0) \neq Rf'(0)$

$\therefore f(x)$ is not diff. at $x = 0$